1. Introduction

Juosan (縦横さん in Japanese) is a puzzle developed by Nikoli, a Japanese publisher that specializes in pencil-and-paper logic puzzles. The puzzle—introduced in 2014 [1]—has been proven NP-complete in 2018 by Iwamoto and Ibusuki [2]. Later in 2020, it was shown that the 3SAT problem is polynomial-time reducible to Juosan puzzles [3]. An algorithmic investigation regarding the card-based zero-knowledge proof of Juosan puzzles is discussed in [4]. This puzzle is played on an $m \times n$ grid of cells that are divided into several rectangular territories enclosed by bold lines. The goal is to fill each cell of the grid with a (vertical bar) or (horizontal bar) symbol following a set of rules, namely [5]:

1. if a territory contains a number, then the number of either or symbols in it must be equal to that number; if a territory is unnumbered, then it may have any number of or symbols;
2. the symbol can extend for more than three cells vertically but not more than two cells horizontally;
3. the symbol can extend for more than three cells horizontally but not more than two cells vertically.

An example of a Juosan puzzle of size 5×5 with seven territories is given in Figure 1. In this example, some territories have constraint numbers.

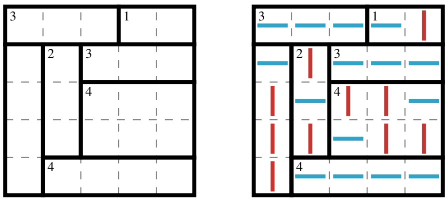

Figure 1. A Juosan puzzle on the left with its respective solution on the right.

Puzzles have a long history and are found in diverse societies, revealing the inherent connection between human intelligence and playful imagination. They serve not only as recreational activities but also as cognitively stimulating exercises [6]. Interestingly, puzzles have gained significant attention from the scientific community in recent decades due to their deep connection with important problems in mathematics and computation [7]. Studies regarding this include algorithmic investigations and computational complexity analysis of various puzzles (see [7–10] for extensive surveys). In the case of pencil-and-paper puzzles, many have been proven to be NP-complete, such as (in chronological order): Nonogram (1996) [11], Sudoku (2003) [12], Nurikabe (2004) [13], Heyawake (2007) [14], Country Road and Yajilin (2012) [15], Kurodoko (2012) [16], Sto-Stone (2018) [17], Usowan (2018) [18], Kurotto and Juosan (2018 and 2020) [2, 3], Tatamibari (2020) [19], Yin-Yang (2021) [20], Nurimeizu (2022) [21], and Tilepaint (2022) [22]. Furthermore, elementary algorithmic investigations have also been conducted on puzzles like Yin Yang and Tatamibari [23, 24].

The NP-completeness of Juosan puzzles implies the existence of a polynomial-time solution verifier and an exponential-time algorithm for solving such puzzles. However, there has been limited formal algorithmic investigation of Juosan solvers because the puzzle is relatively new and has only been recently proven NP-complete. There are various approaches to solving NP-complete puzzles, including the utilization of MIP (mixed integer programming) solvers, SAT-based solvers, or using an elementary backtracking approach [10, 25, 26]. This paper delves into two techniques: the backtracking method with pruning optimizations and a SAT-based approach. Furthermore, a comparative analysis of their performance in solving a range of Juosan puzzles is presented. The findings also demonstrate that the backtracking technique can solve any Juosan puzzle in exponential time relative to its size.

As for the SAT-based approach, an efficient SAT encoding of the Juosan puzzle is introduced, allowing for effective utilization within a SAT solver. Numerous studies have showcased the effectiveness of SAT solvers in tackling NP-complete puzzles, such as Sudoku [26–30], Edge Matching puzzles [31], Rubik's cube [32], Binary puzzles [33], Fill-a-pix puzzles [34], and Skyscrapers puzzles [35].

This paper also discusses some special tractable cases of the Juosan puzzle and pertinent mathematical analyses regarding the number of solutions. Investigation of tractable sub-problems and tractable variants of NP-complete problems are particularly important in theoretical computer science (see, e.g., [36, 37]). Moreover, counting the number of solutions to a particular computational problem is also interesting from mathematical and computational perspectives, especially in counting complexity theory [38, 39].

The rest of the paper is further organized into the following sections. Section 2 discusses some theoretical aspects of the Juosan puzzles and derives an additional rule concerning the non-existence of certain subgrids.

In Section 3, an $O(mn)$ time algorithm is introduced to verify an $m \times n$ Juosan solution. Section 4 presents an optimized backtracking algorithm for solving an arbitrary $m \times n$ Juosan puzzle in $O(2^{mn})$ time. Section 5 presents a SAT-based approach for solving Juosan puzzles, including an in-depth analysis of the number of variables and associated clauses. Section 6 discusses the investigation of tractable variants of the Juosan puzzle and the upper bound on their number of solutions based on mathematical analysis. Additionally, computational experiments of the algorithms are presented in Section 7. The last portion of the paper, Section **??**, summarizes and concludes the findings.